

CSE 250

Data Structures

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Day 23
Priority Queues, Heaps
Textbook Ch. 18

Announcements

- WA2 released, due 10/31
- PA3 has not been released yet
- Midterm exam grades have been posted

PriorityQueue ADT

PriorityQueue [A <: Ordering]

enqueue (v: A) : Unit

Insert value **v** into the priority queue

dequeue : A

Remove the greatest element in the priority queue

head : A

Peek at the greatest element in the priority queue

Priority Queues

Two mentalities...

Lazy: Keep everything a mess ("Selection Sort")

Proactive: Keep everything organized ("Insertion Sort")

Priority Queues

Operation	Lazy	Proactive
enqueue	$O(1)$	$O(n)$
dequeue	$O(n)$	$O(1)$
head	$O(n)$	$O(1)$

Priority Queues

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Can we do better?

Priority Queues

Lazy - Fast Enqueue, Slow Dequeue

Proactive - Slow Enqueue, Fast Dequeue

Priority Queues

Lazy - Fast Enqueue, Slow Dequeue

Proactive - Slow Enqueue, Fast Dequeue

??? - Fast(-ish) Enqueue, Fast(-ish) Dequeue

Priority Queues

Idea: Keep the priority queue "kinda" sorted.

Hopefully "kinda" sorted is cheaper to maintain than a full sort,
but still gives us some of the benefits.

Priority Queues

Idea: Keep the priority queue "kinda" sorted.

Keep larger items towards the front of the list,
and keep the front of the list more sorted than the back...

Binary Heaps

Challenge: If we are only "kinda" sorting, how do we know which elements are actually sorted?

Binary Heaps

Idea: Organize the priority queue as a *directed* tree!

A directed edge from ***a*** to ***b*** means that **$a \geq b$**

More Tree Terminology

Child - An adjacent node connected by an out-edge

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Depth (of a tree) - The maximum depth of any node in the tree

Level (of a node) - $\text{depth} + 1$

More Tree Terminology

A is the root

B and **C** are children of **A**

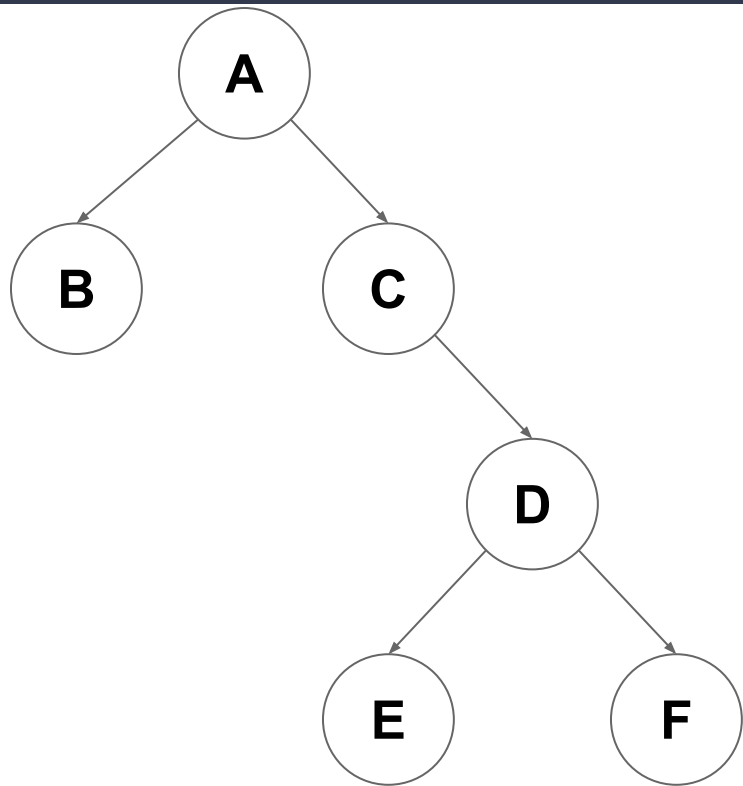
D is a child of **C**

E and **F** are children of **D**

B, **E** and **F** are leaves

The depth of **A** is 0, **B** and **C**: 1, **D**: 2, **E** and **F**: 3

The depth of the tree is 3



Binary Heaps

Organize our priority queue as a directed tree

Directed: A directed edge from a to b means that $a \geq b$

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Complete: Every "level" except the last is full (from left to right)

Binary Heaps

Organize our priority queue as a directed tree

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Balanced: TBD (basically, all leaves are roughly at the same level)

Binary Heaps

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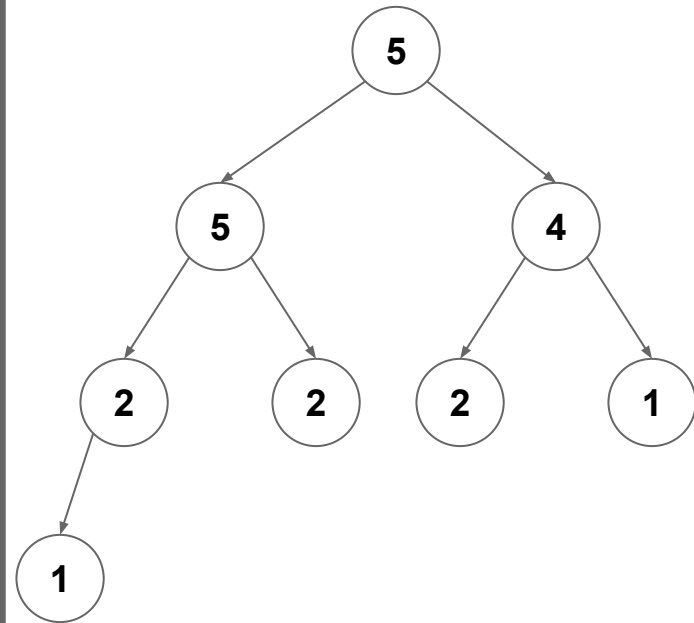
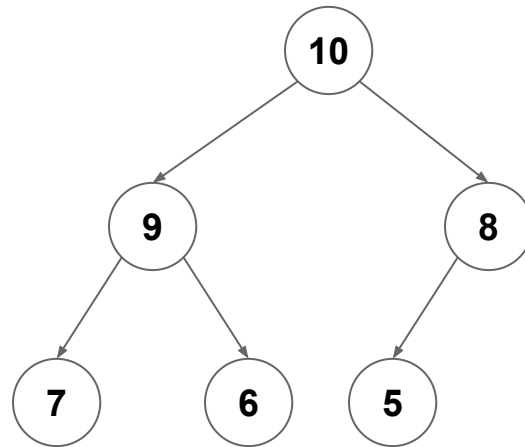
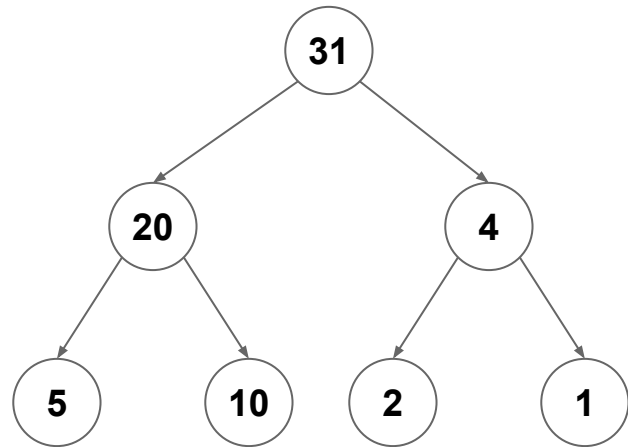
Balanced: TBD (basically, all leaves are roughly at the same level)

This makes it easy to encode into an array (later today)

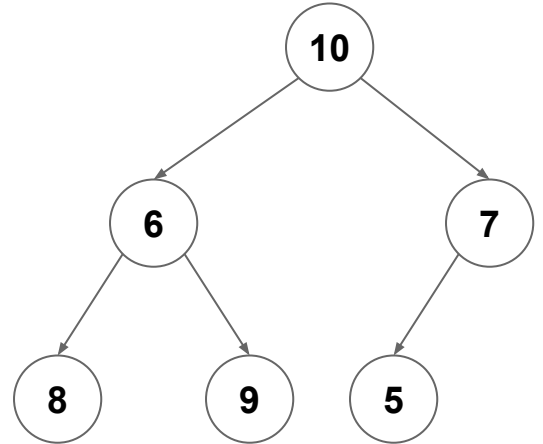
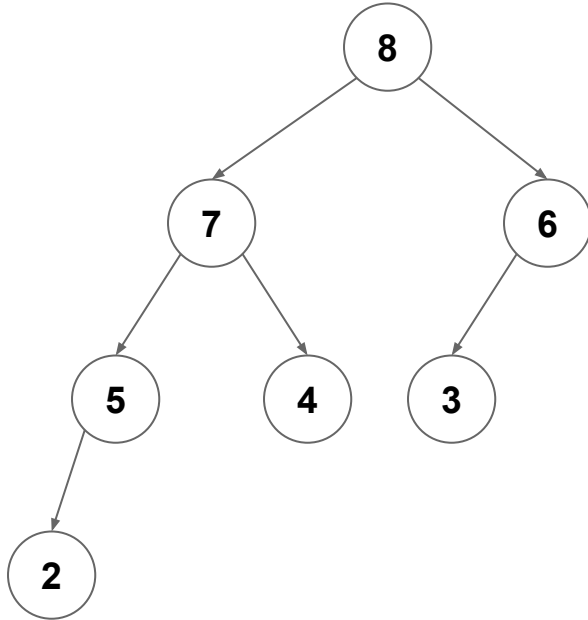
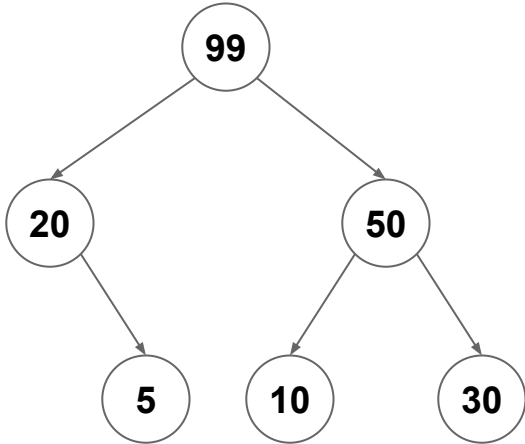
Max Heaps

If we use \geq as our ordering operation, we have a Max Heap
(as compared to a Min Heap)

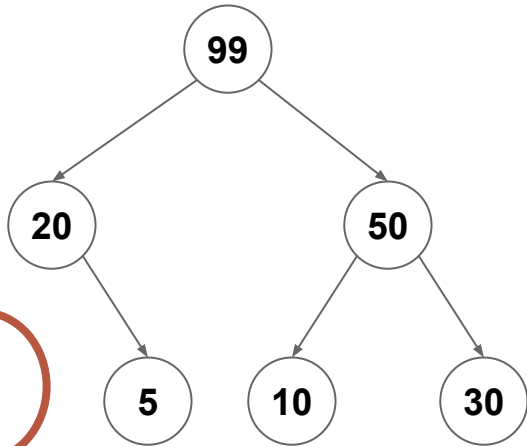
Valid Max Heaps



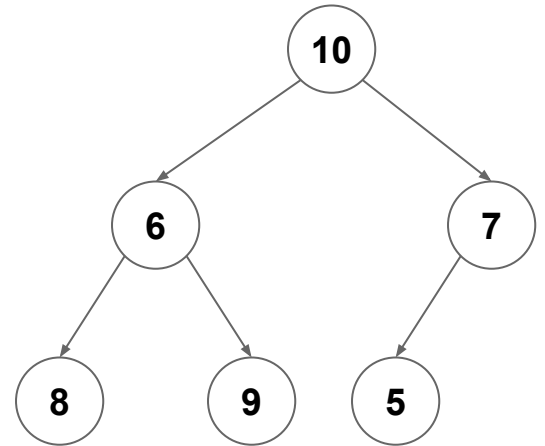
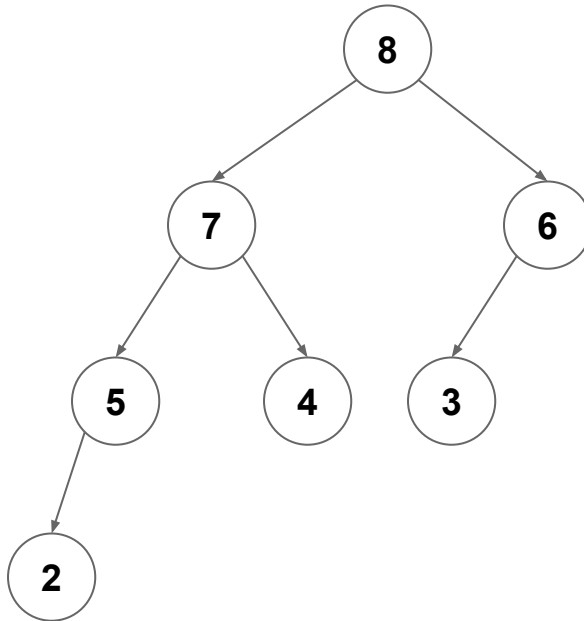
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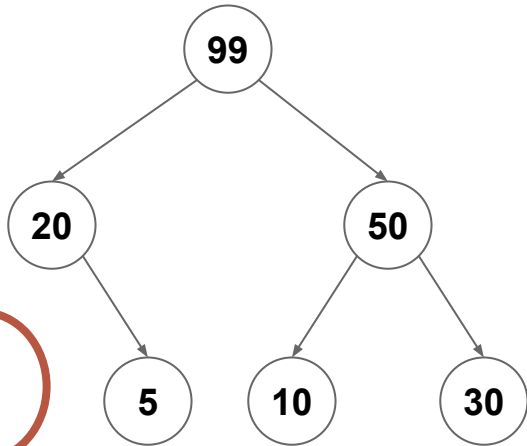
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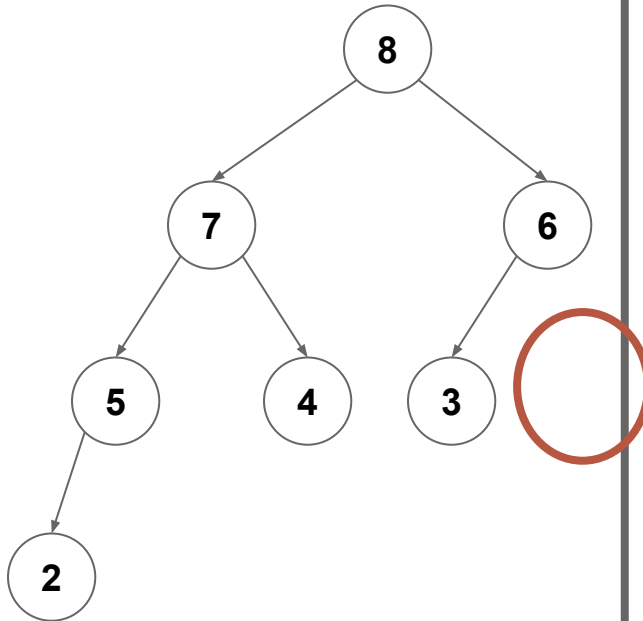
Need to fill from left to right



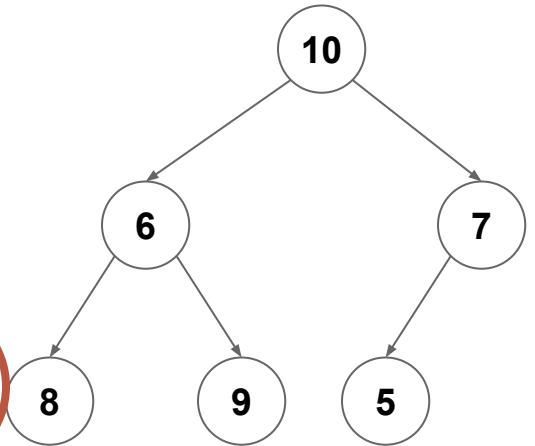
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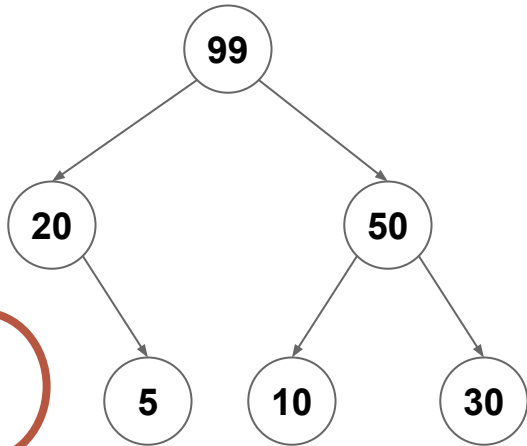
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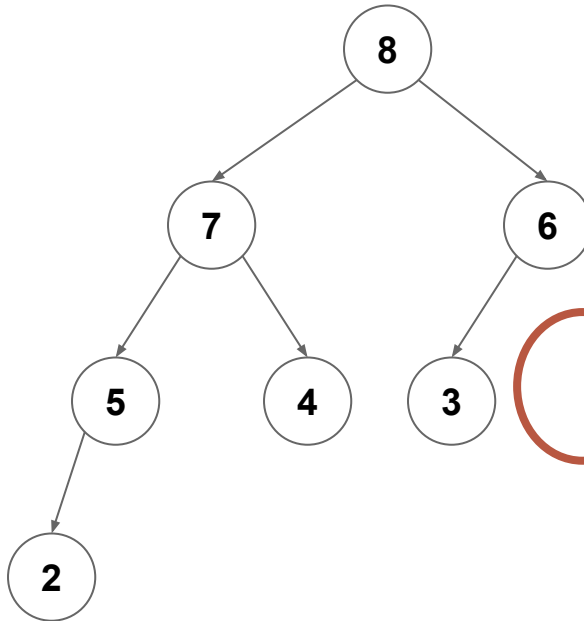
Need complete levels



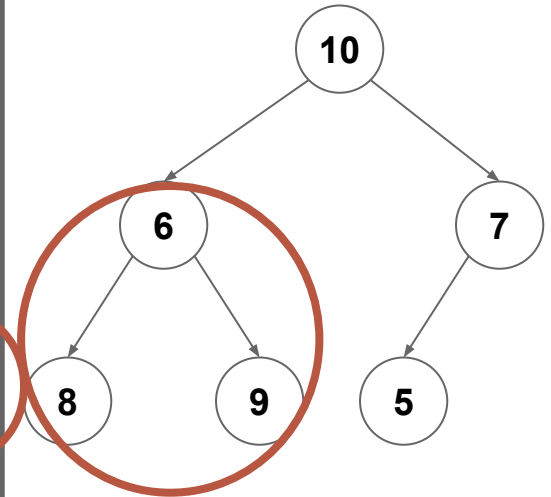
Invalid Max Heaps



Need to fill from left to right



Need complete levels



Children must be less than or equal to parents

Heaps

What is the depth of a binary heap containing n items?

Level 1: holds up to 1 item

Heaps

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Level 3: holds up to 4 items

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What is the depth of a binary heap containing n items?

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Level 4: holds up to 8 items

Heaps

What is the depth of a binary heap containing n items?

Level 1: holds up to 1 item

Level 2: holds up to 2 items

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Level 4: holds up to 8 items

...

Level i : holds up to 2^{i-1} items

Heaps

What is the depth of a binary heap containing n items?

$$n = O \left(\sum_{i=1}^{\ell_{max}} 2^i \right) = O(2^{\ell_{max}})$$

Heaps

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$$\ell_{max} = O(\log(n))$$

The Heap ADT

enqueue (elem: A) : Unit *[AKA pushHeap]*
Place an item into the heap

dequeue : A *[AKA popHeap]*
Remove and return the maximal element from the heap

head : A
Peek at the maximal element in the heap

length : Int
The number of elements in the heap

Heap . enqueue

Idea: Insert the element at the next available spot, then fix the heap.

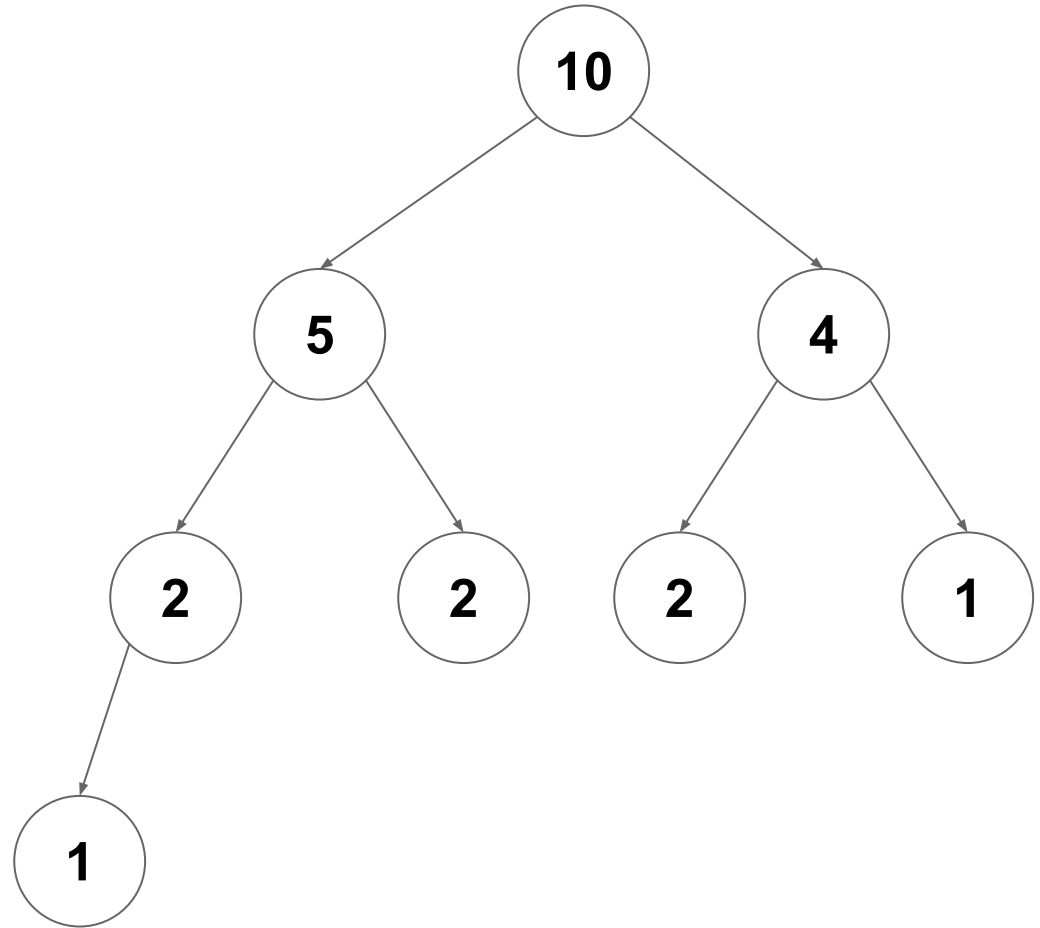
Heap.enqueue

Idea: Insert the element at the next available spot, then fix the heap.

1. Call the insertion point `current`
2. While `current != root` and `current > parent`
 - a. Swap `current` with `parent`
 - b. Repeat with `current ← parent`

Heap . enqueue

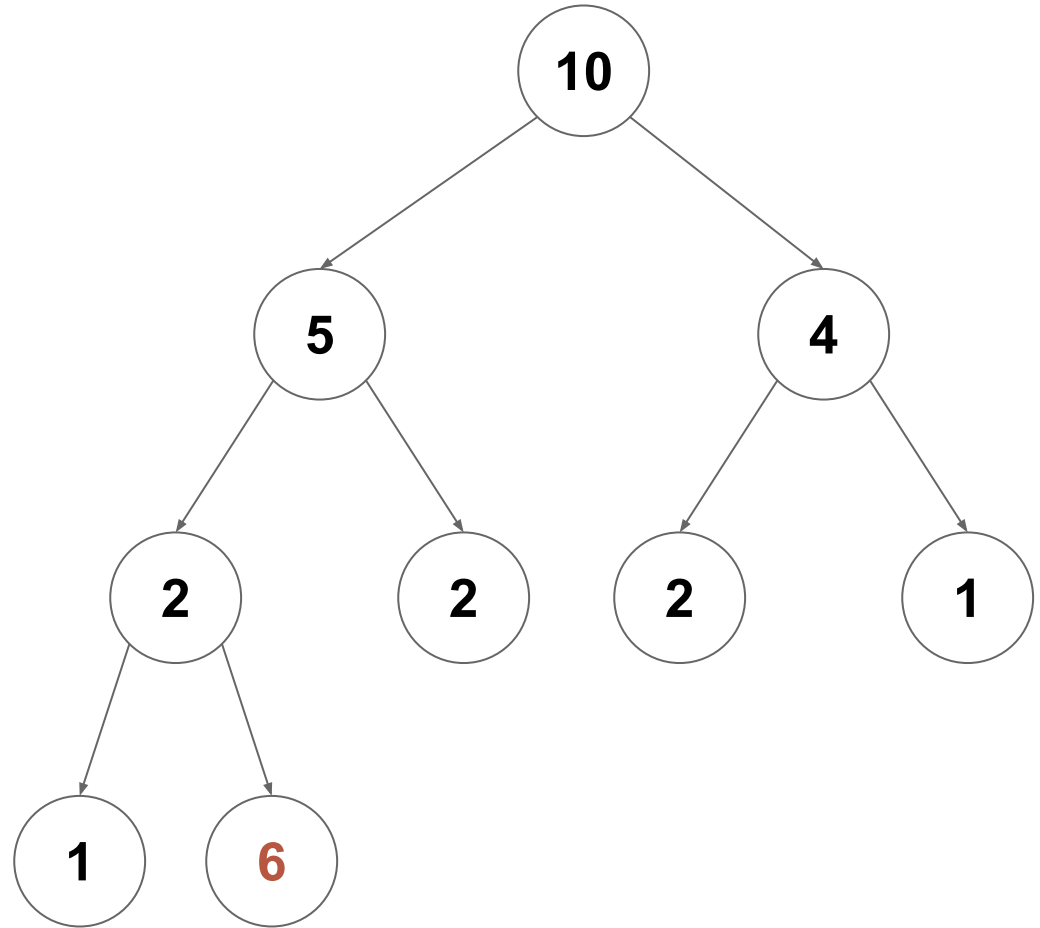
What if we enqueue 6?



Heap . enqueue

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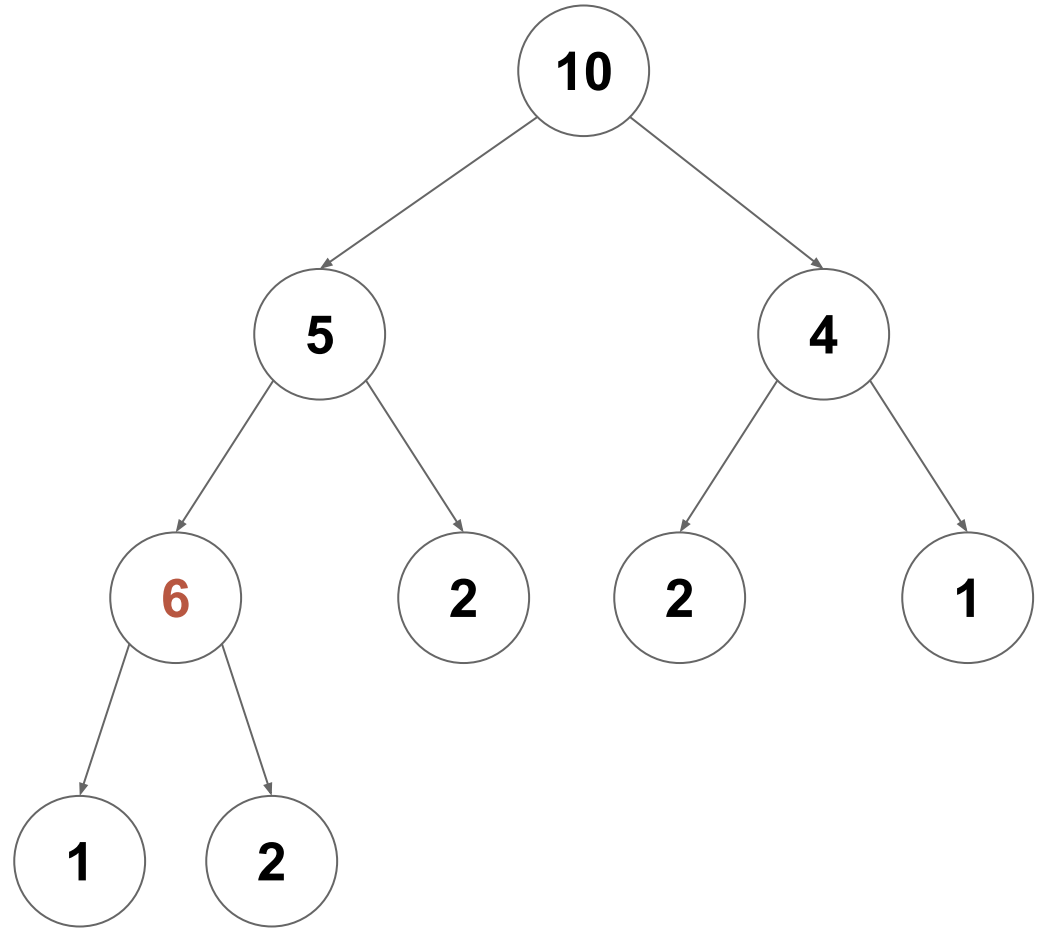
Place in the next available spot



Heap . enqueue

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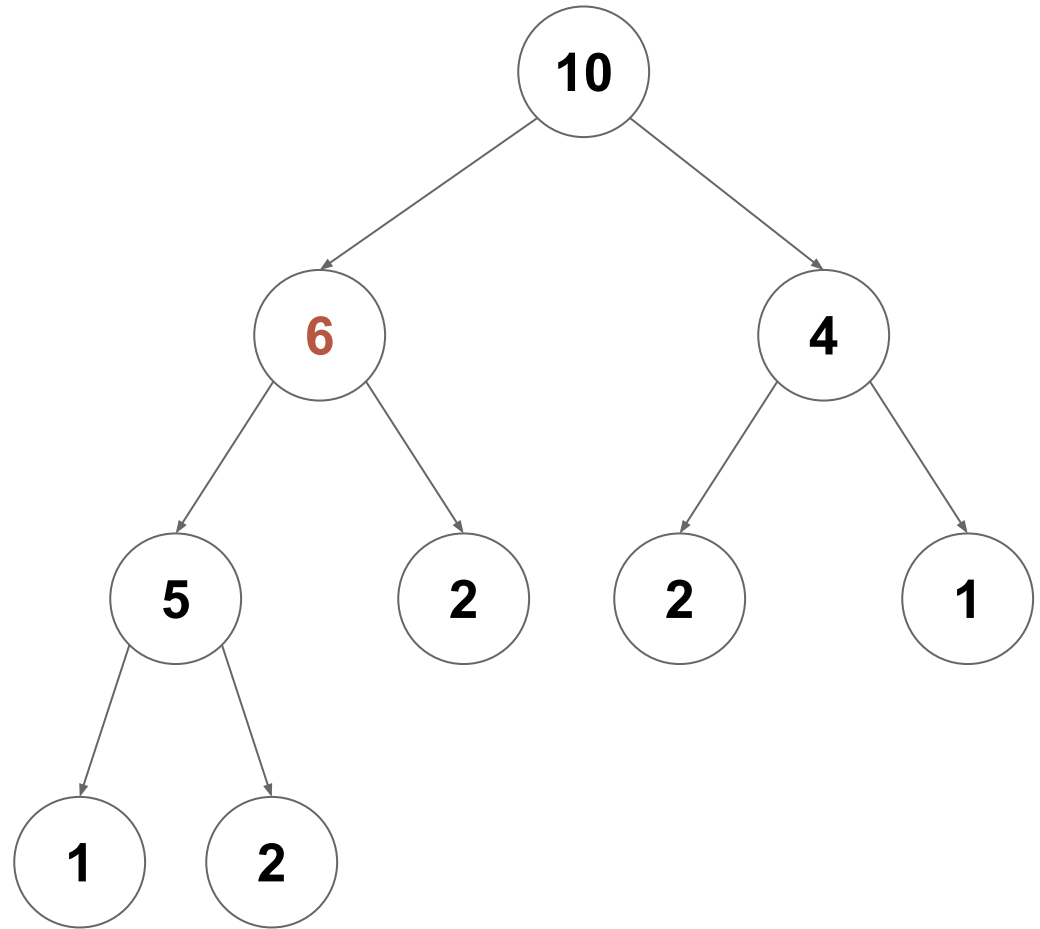
Swap with parent if it is bigger than the parent



Heap . enqueue

What if we enqueue 6?

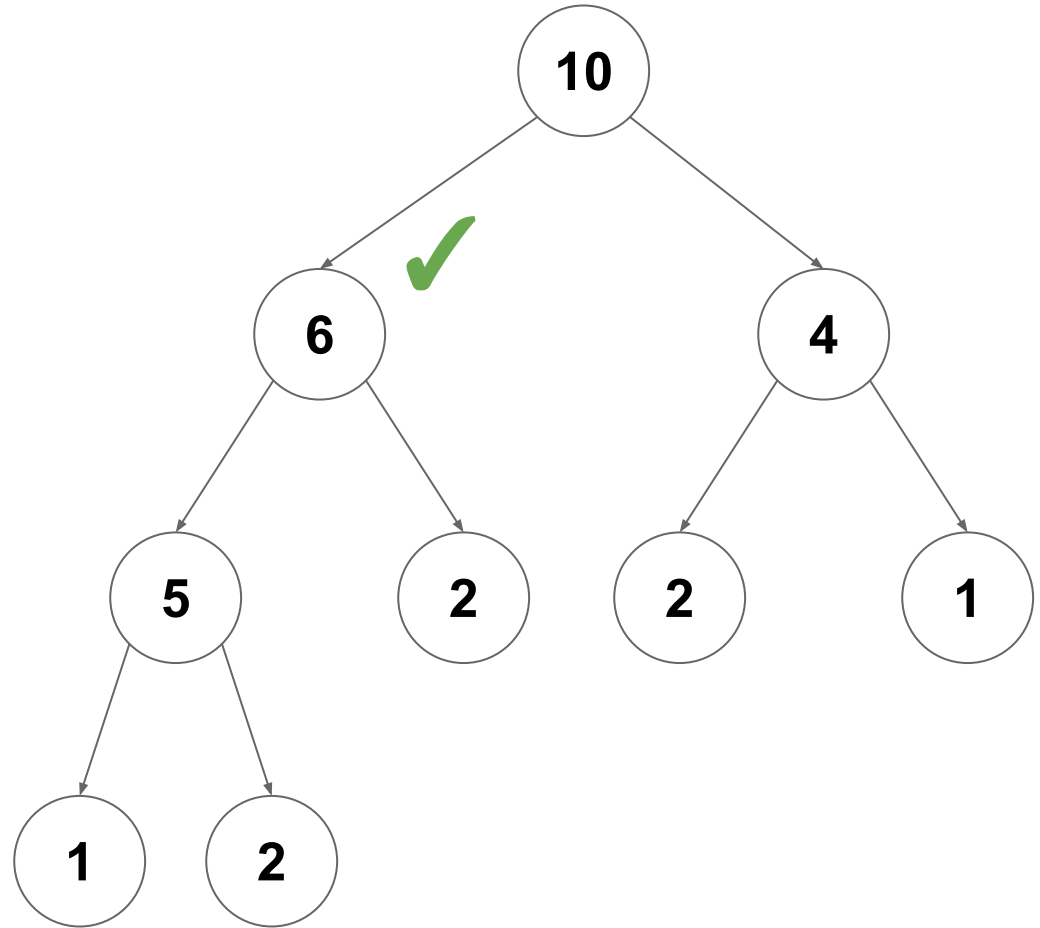
Continue swapping
upwards...



Heap . enqueue

What if we enqueue 6?

Stop swapping when we are no longer bigger than our parent



Heap.enqueue - fixUp

```
def fixUp[A: Ordering](current: Vertex[A]): Unit = {  
  if(current.parent.isDefined){  
    val parent = current.parent.get  
    if( Ordering[A].lt( parent.value, current.value ) ){  
      swap(current.value, parent.value)  
      fixUp(parent)  
    }  
  }  
}
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What is the complexity (or how many swaps occur)?

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What is the complexity (or how many swaps occur)? $O(\log(n))$

Heap . dequeue

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Idea: Replace root with the last element then fix the heap

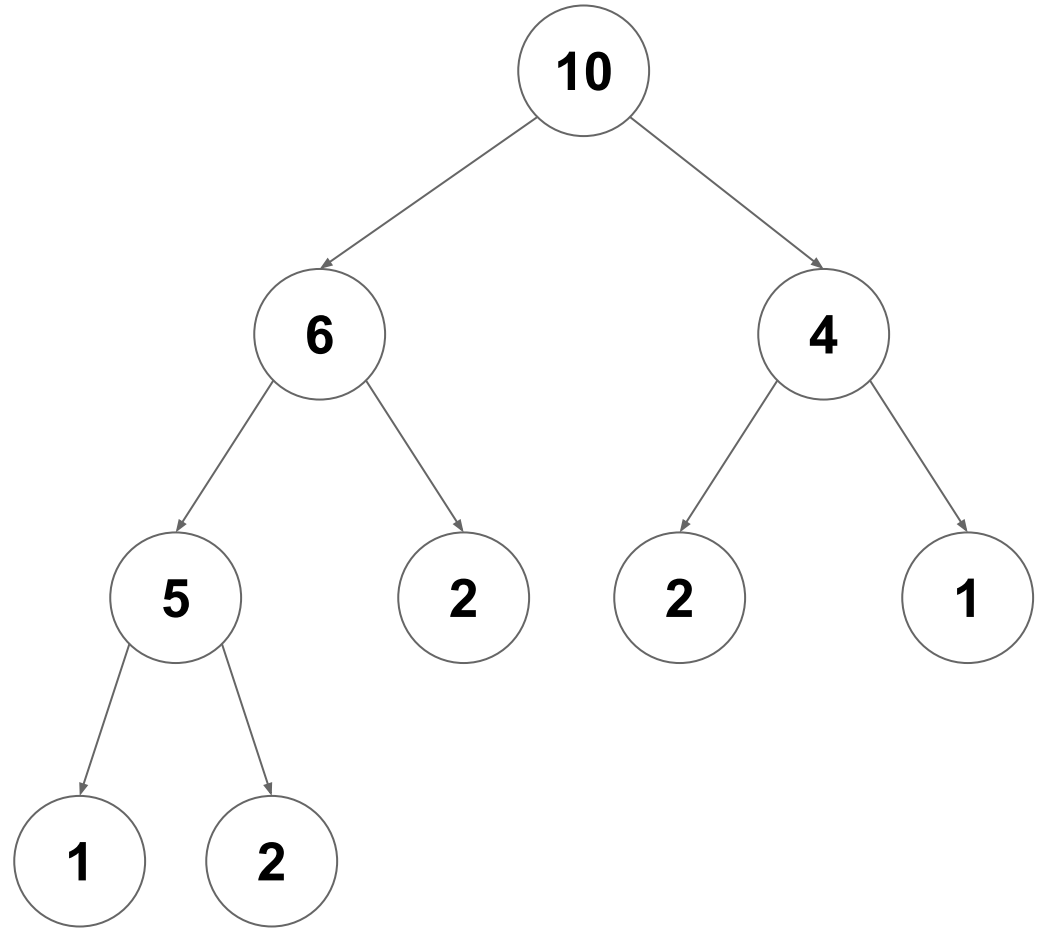
Heap . dequeue

Idea: Replace root with the last element then fix the heap

1. Start with `current ← root`
2. While `current` has a `child > current`
 - a. Swap `current` with its largest `child`
 - b. Repeat with `current ← child`

Heap . dequeue

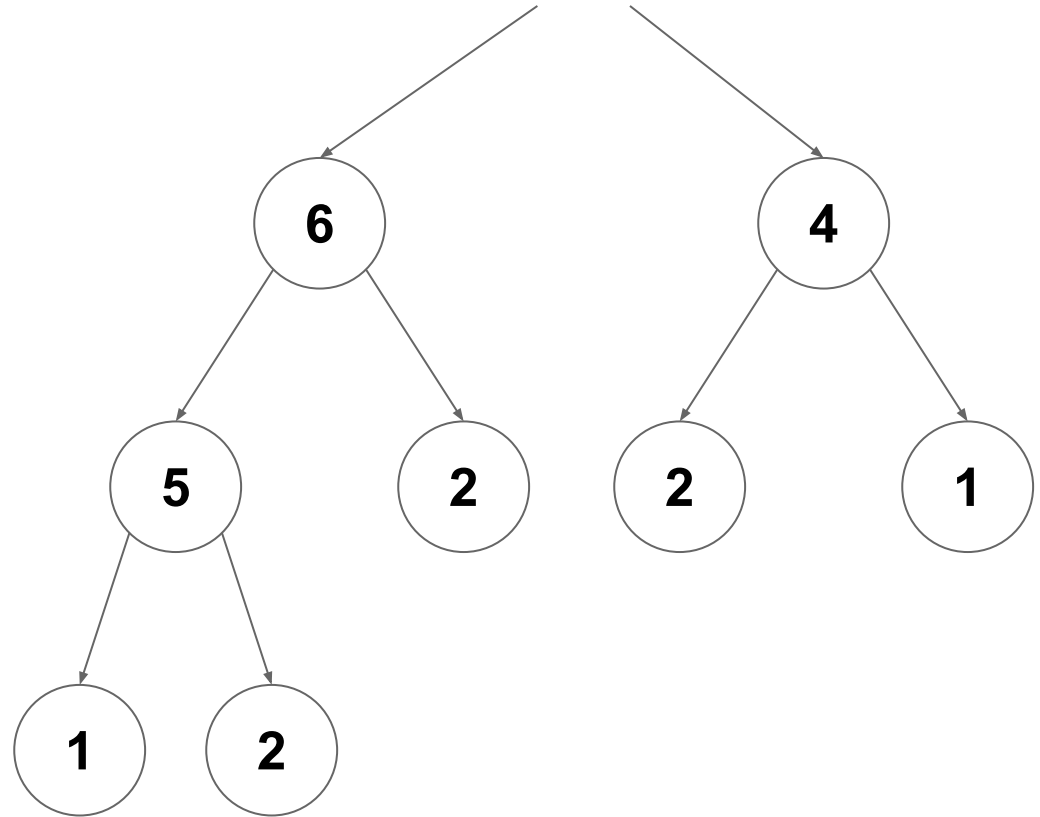
What if we call dequeue?



Heap . dequeue

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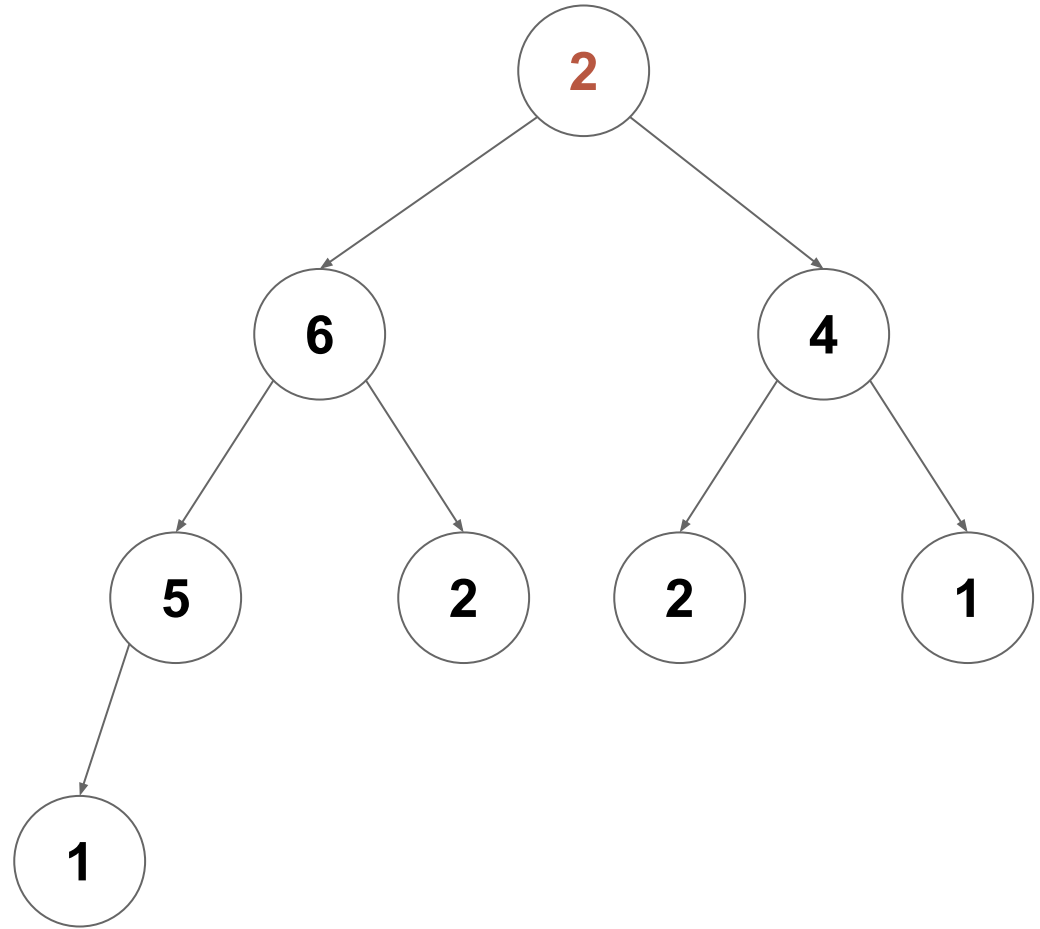
Remove and return the
root



Heap.dequeue

What if we call dequeue?

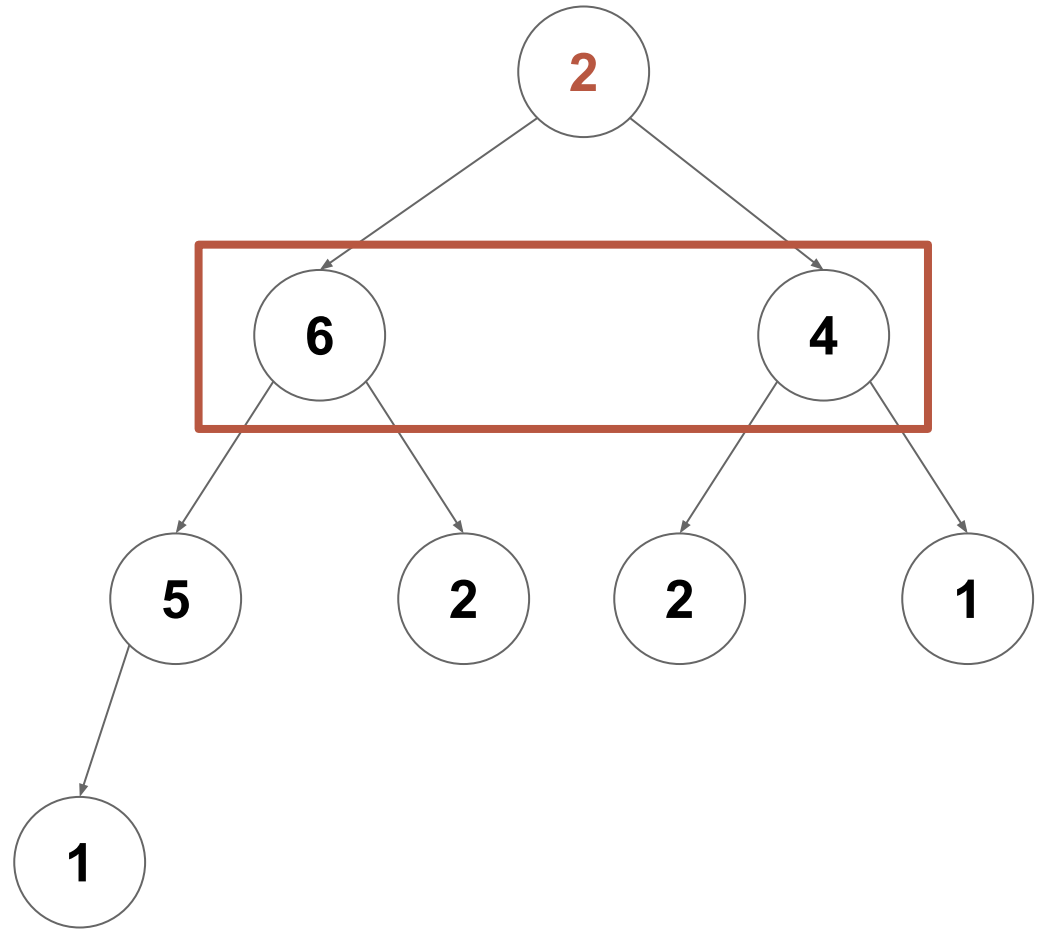
Make the last item the
new root



Heap . dequeue

What if we call dequeue?

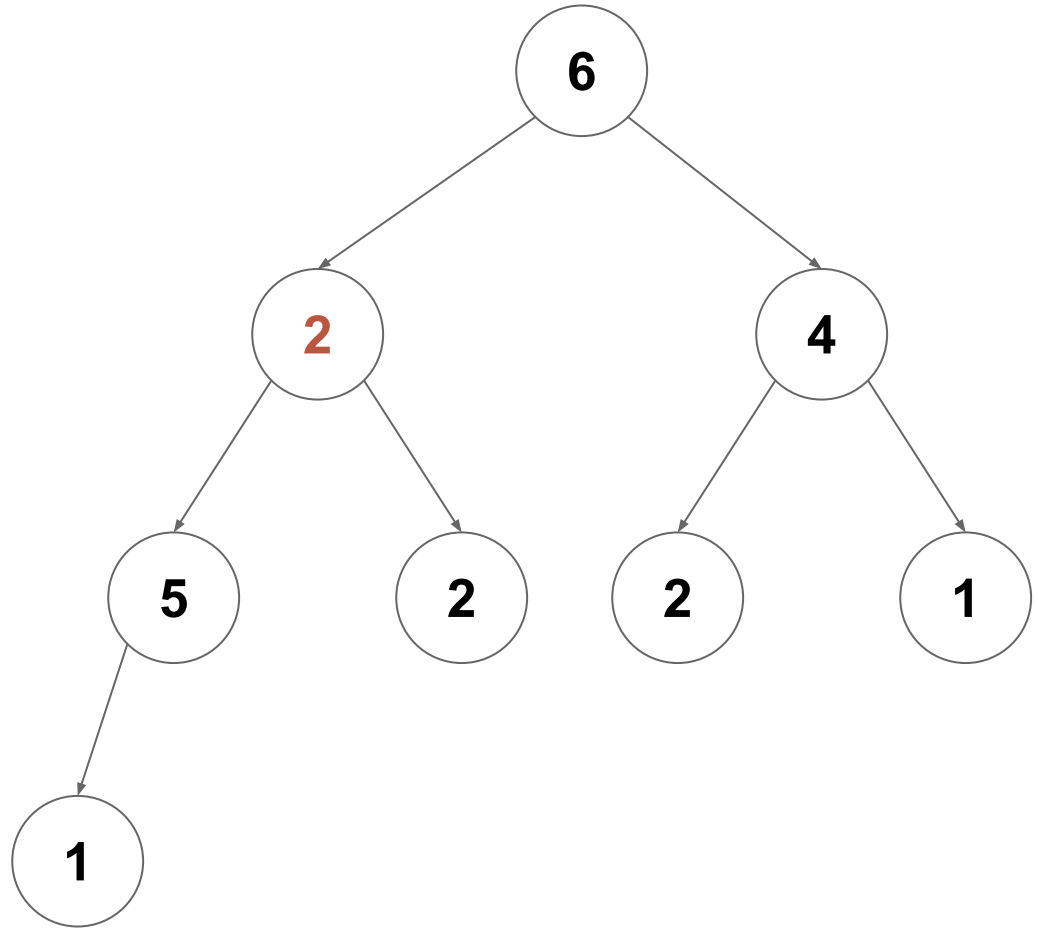
Check for our largest child



Heap . dequeue

What if we call dequeue?

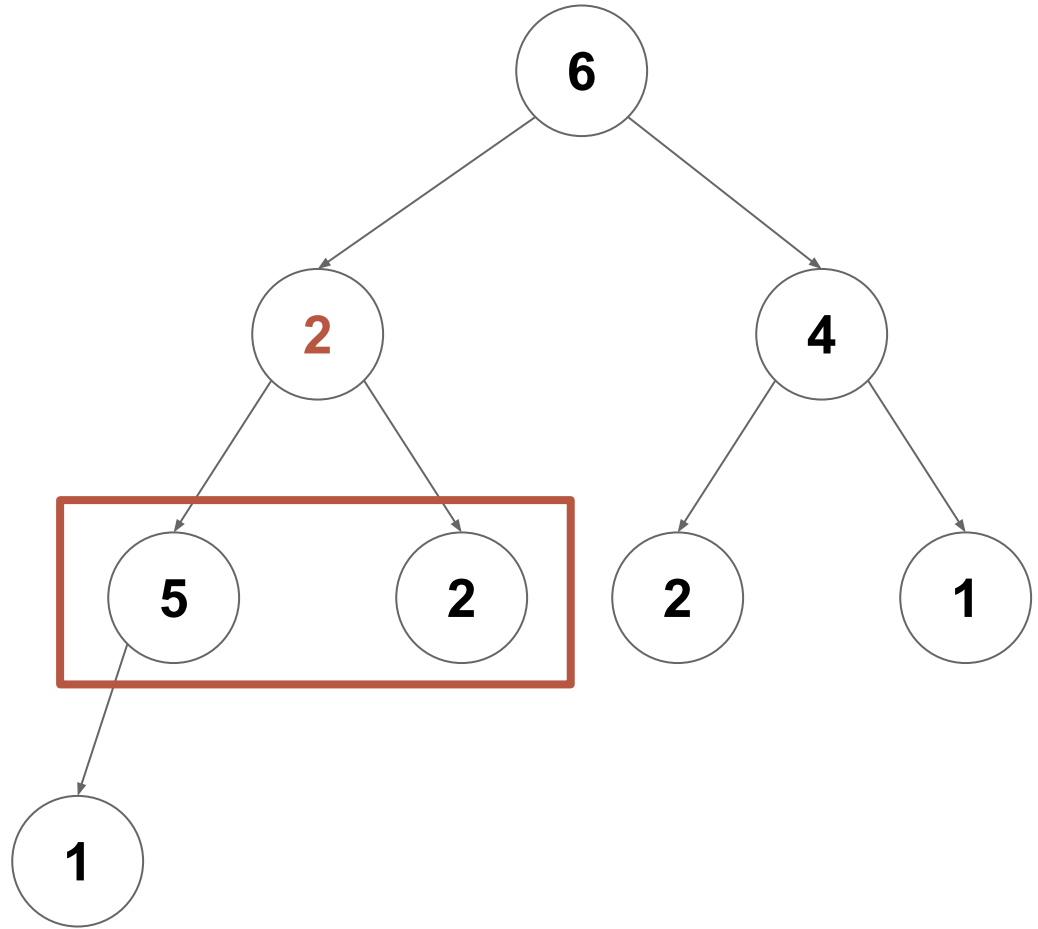
If the largest child is bigger than us, swap



Heap . dequeue

What if we call dequeue?

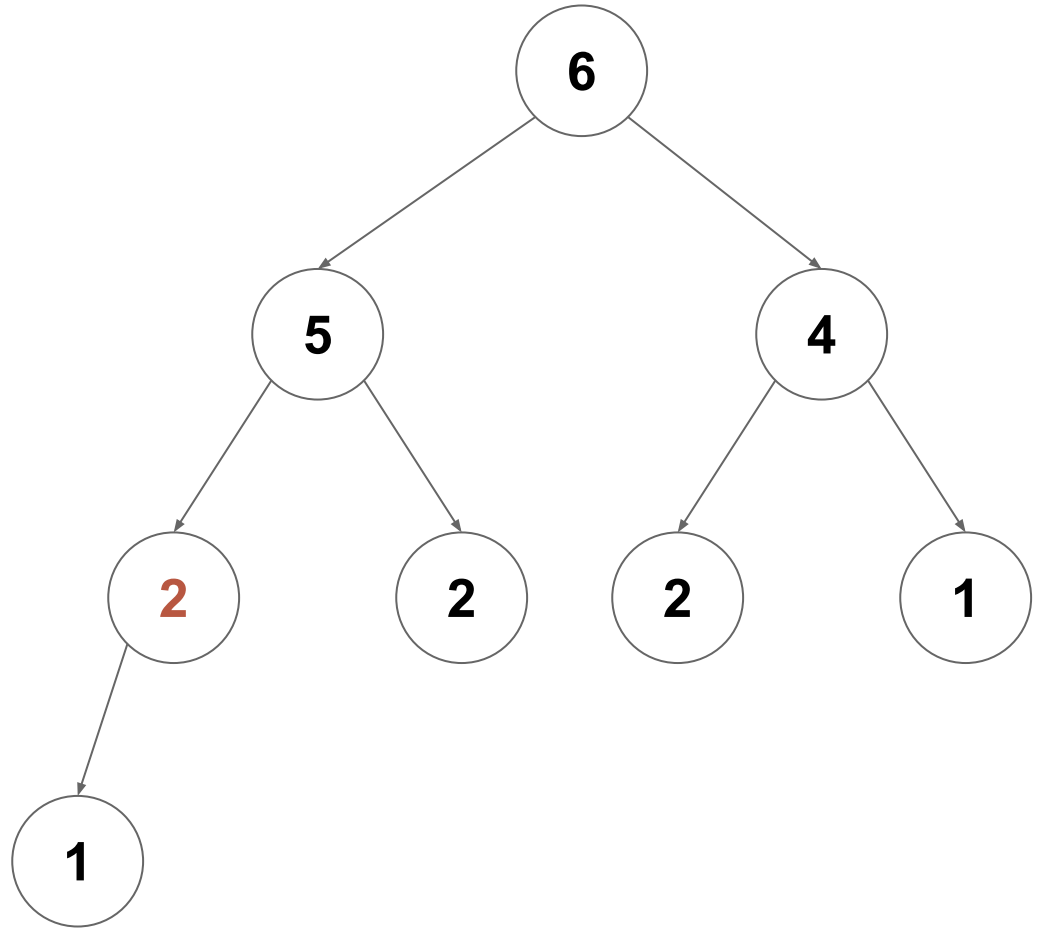
Continue swapping down the tree as necessary...



Heap . dequeue

What if we call dequeue?

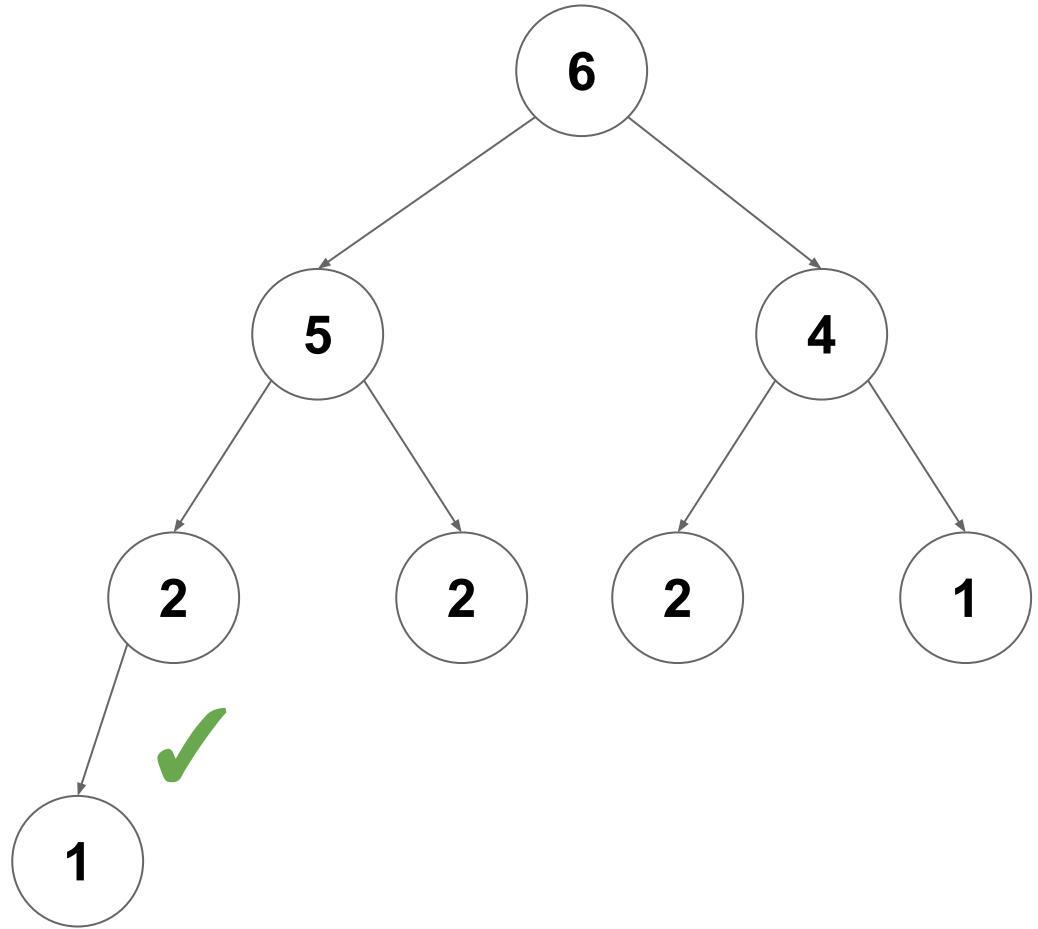
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Heap . dequeue

What if we call dequeue?

Stop swapping when our children are no longer bigger



Heap.dequeue - fixDown

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def fixDown[A: Ordering](current: Vertex[A]): Unit = {  
  val maxChild = getMaxChildOf(current)  
  if( maxChild.isDefined ) {  
    val max = maxChild.get  
    if( Ordering[A].lt( current.value, max.value ) ){  
      swap(current.value, max.value);  
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    }  
  }  
}
```

Heap.dequeue - fixDown

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What is the complexity (or how many swaps occur)? $O(\log(n))$

Priority Queues

Operation	Lazy	Proactive	Heap
enqueue	$O(1)$	$O(n)$	$O(\log(n))$
dequeue	$O(n)$	$O(1)$	$O(\log(n))$
head	$O(n)$	$O(1)$	$O(1)$

Storing heaps

Notice that:

1. Each level has a maximum size
2. Each level grows left-to-right
3. Only the last layer grows

How can we compactly store a heap?

Storing heaps

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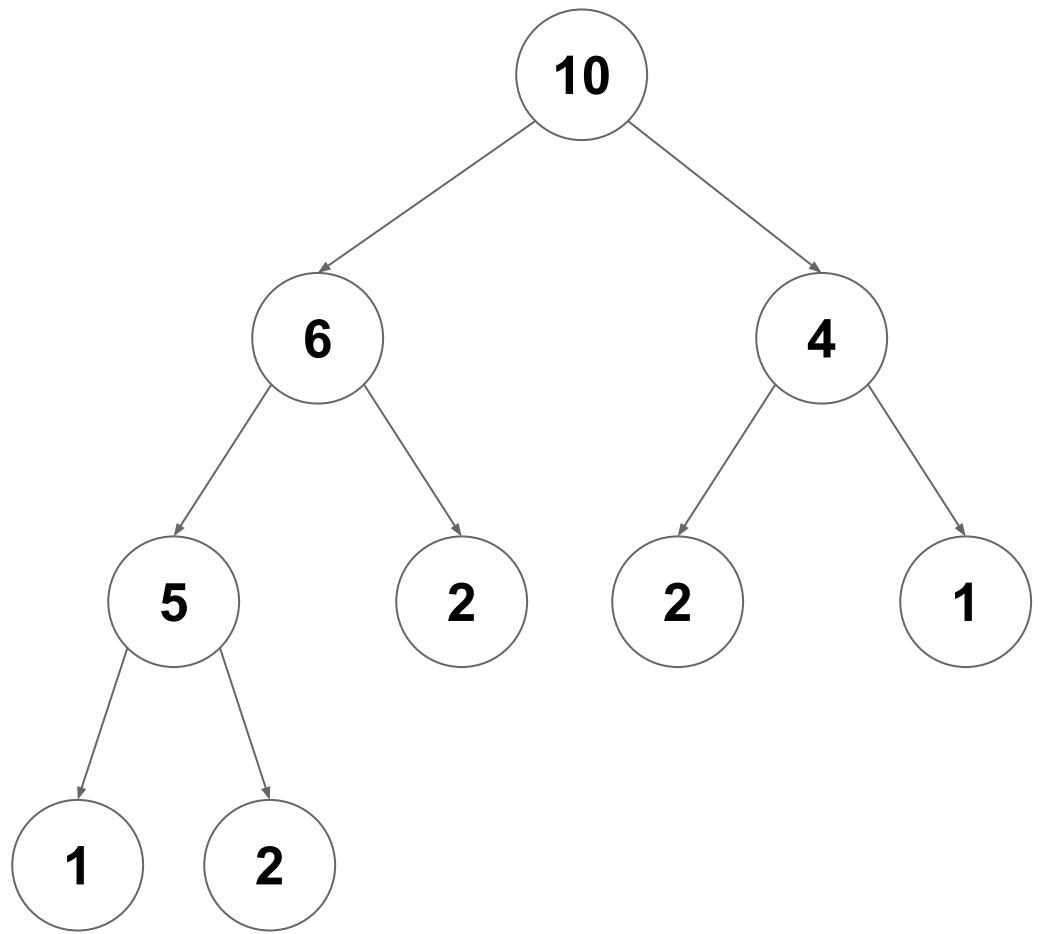
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How can we compactly store a heap?

Idea: Use an `ArrayBuffer`

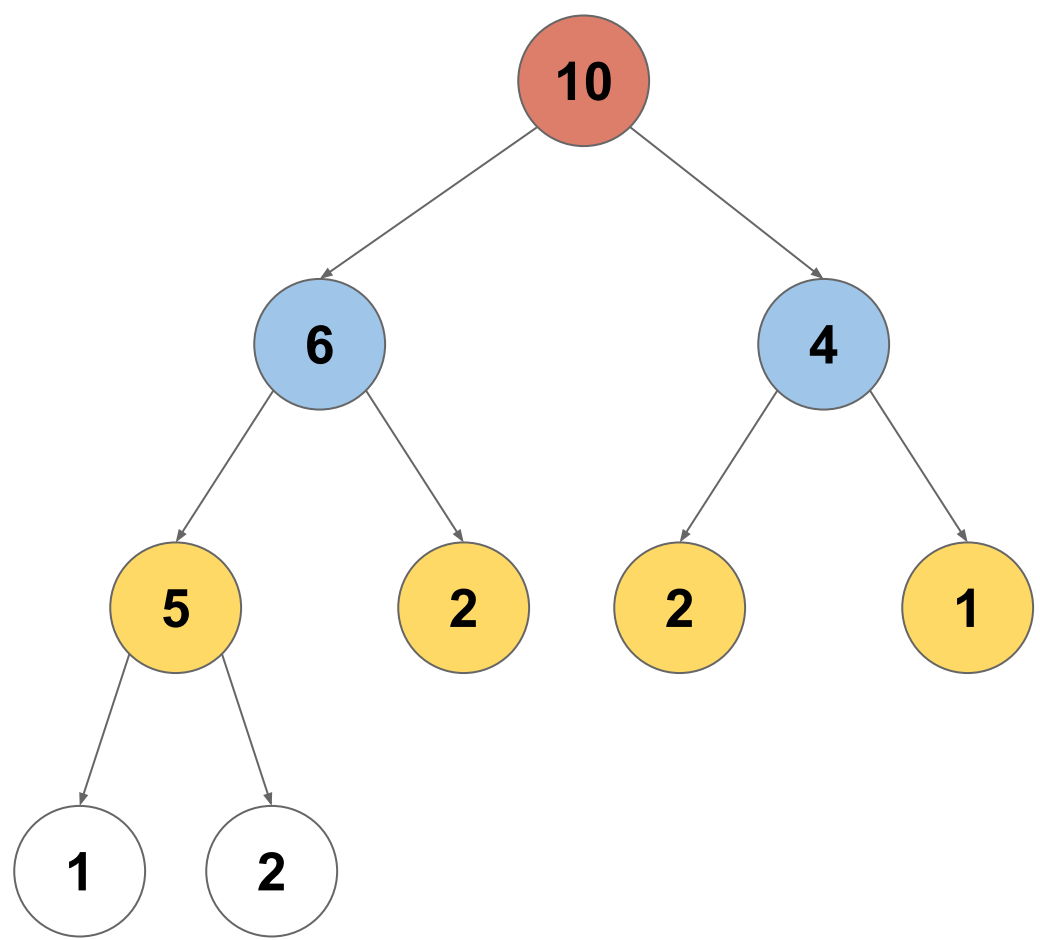
Storing Heaps

How can we store this heap in an array buffer?



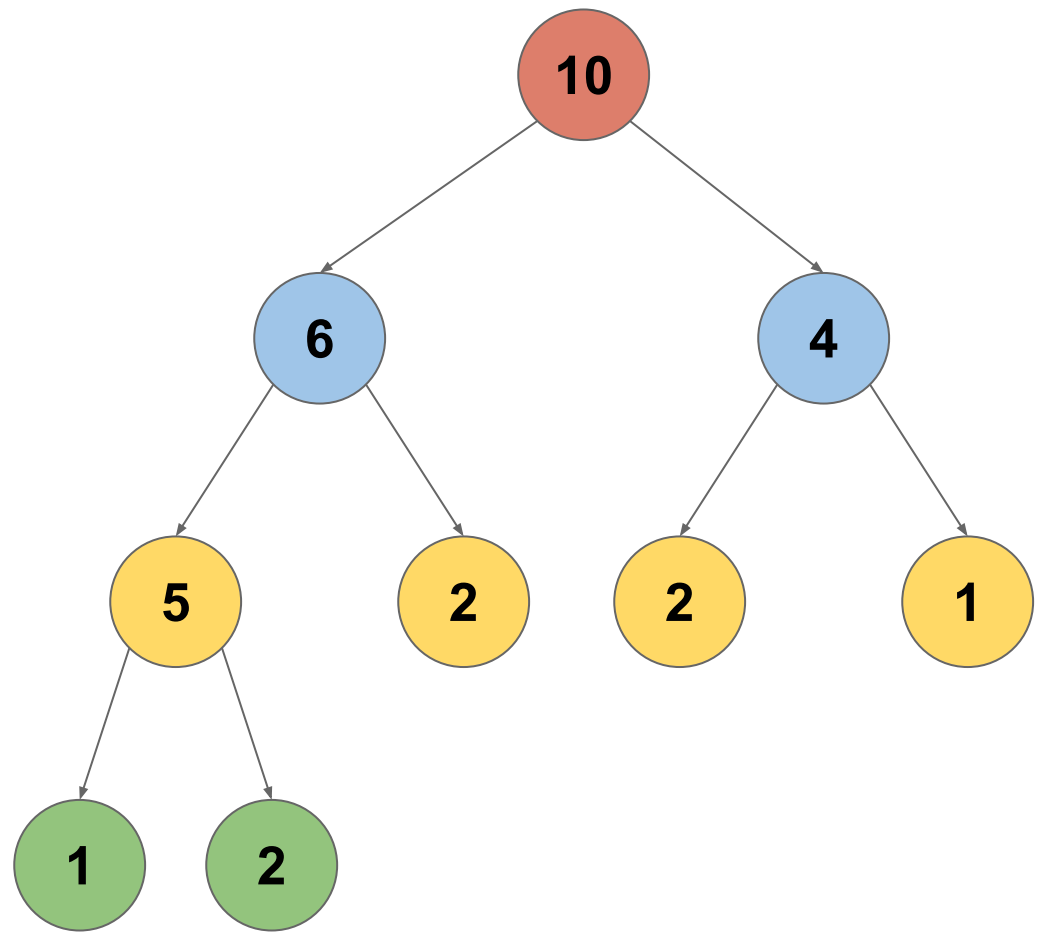
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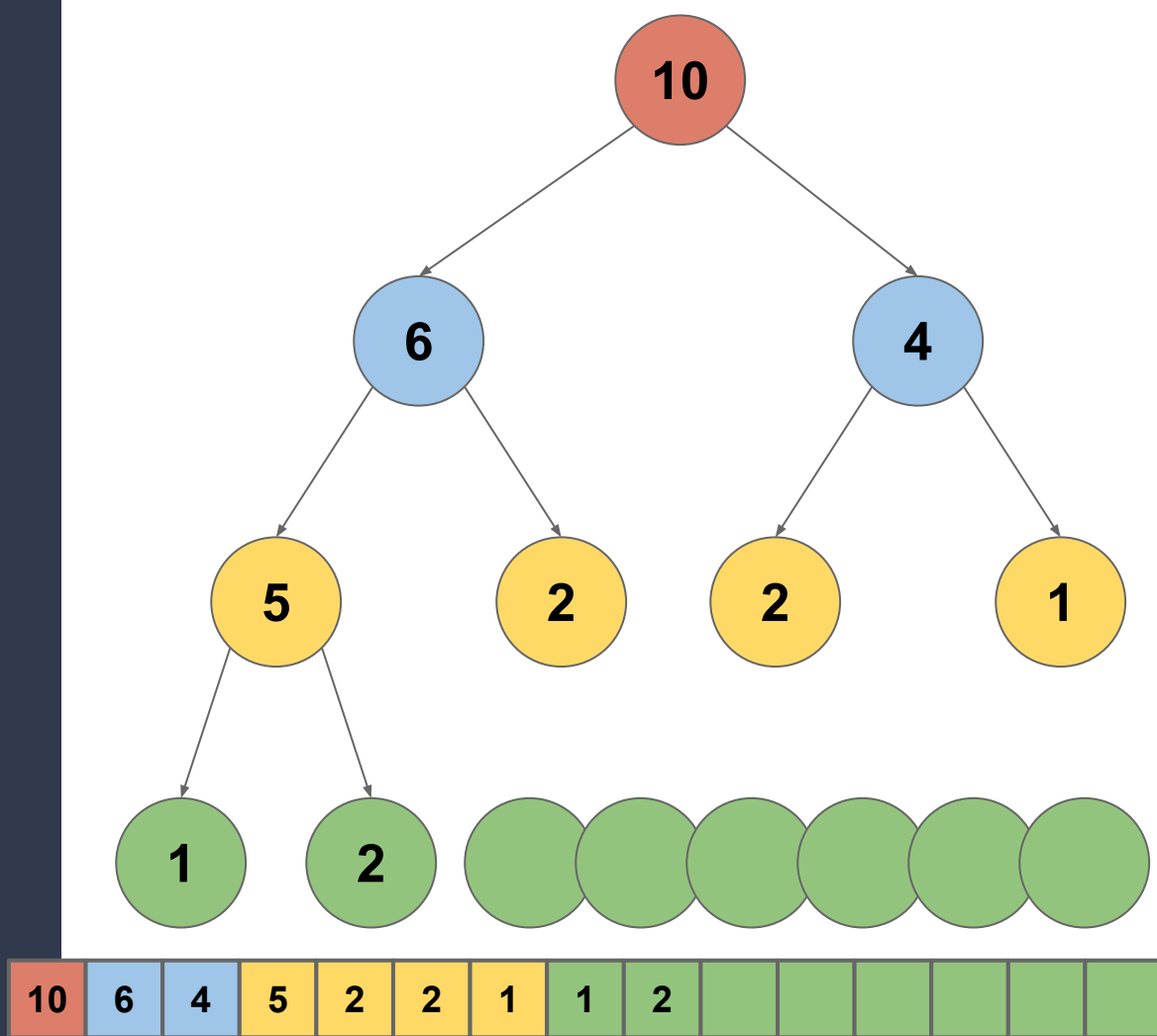
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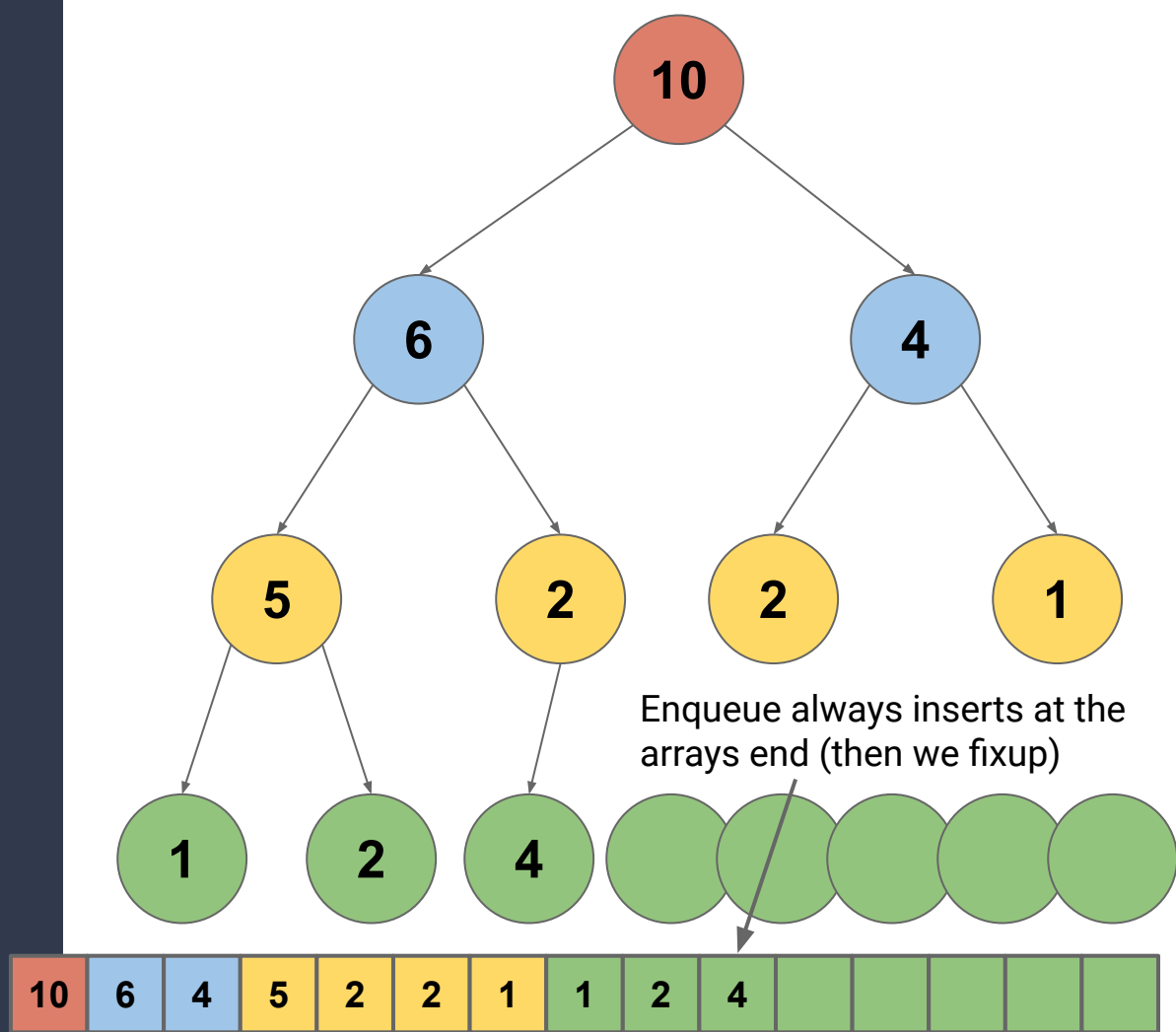
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Runtime Analysis

`enqueue`

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enqueue

- **Append to `ArrayBuffer`:** amortized $O(1)$ (*worst-case $O(n)$*)

Runtime Analysis

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dequeue

- **Remove end of ArrayBuffer:** $O(1)$

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Heap Sort

1. Insert items into heap
2. Reconstruct sequence (in reverse order) with dequeue

7, 4, 8, 2, 5, 3, 9



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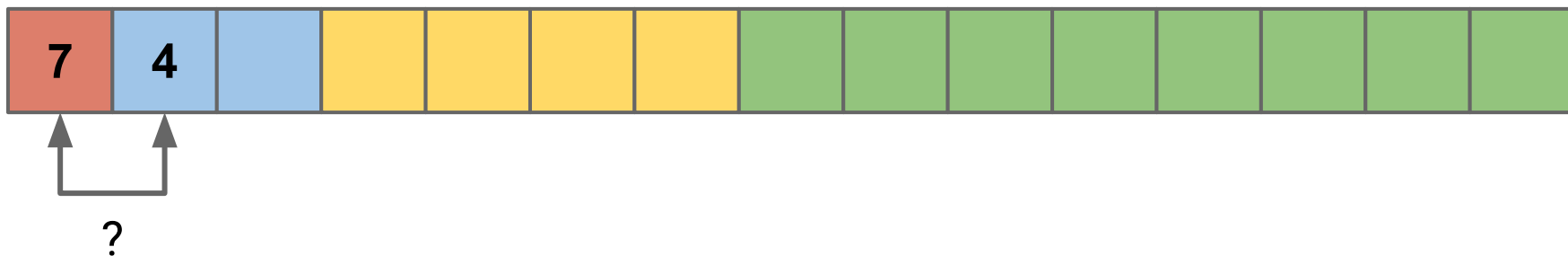
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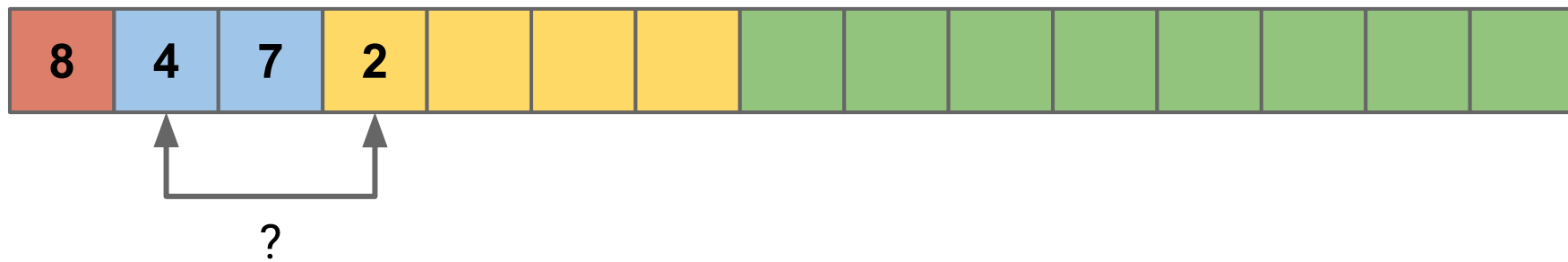
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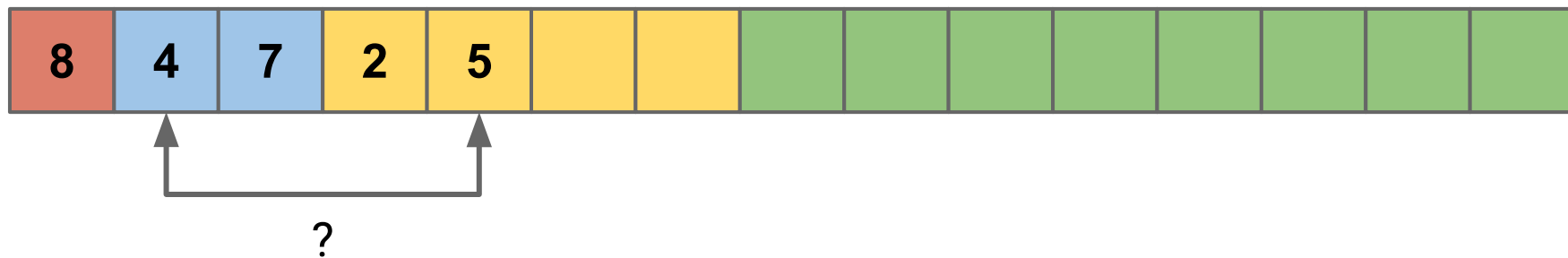
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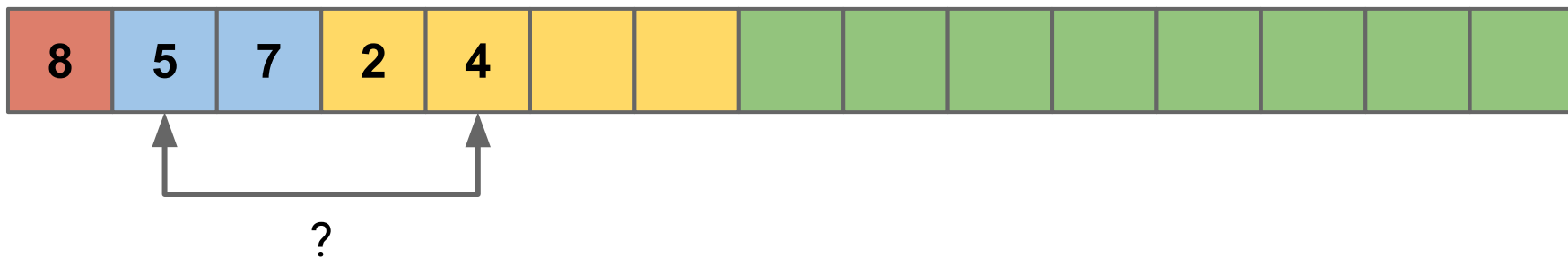
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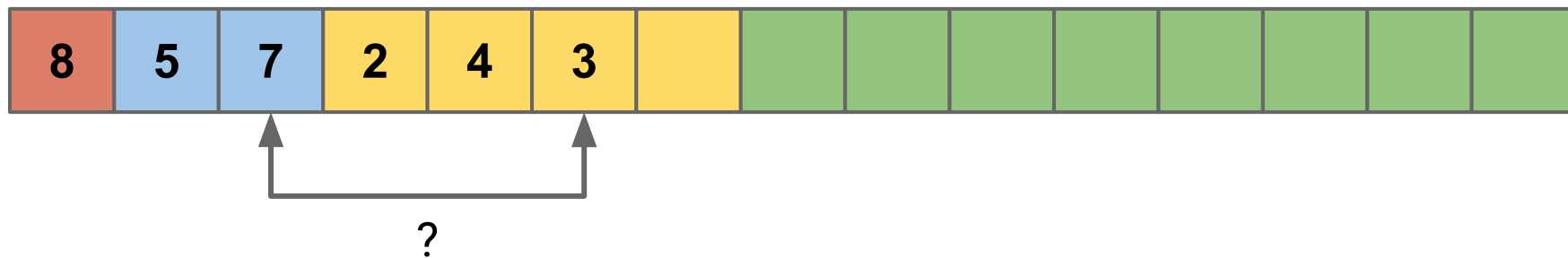
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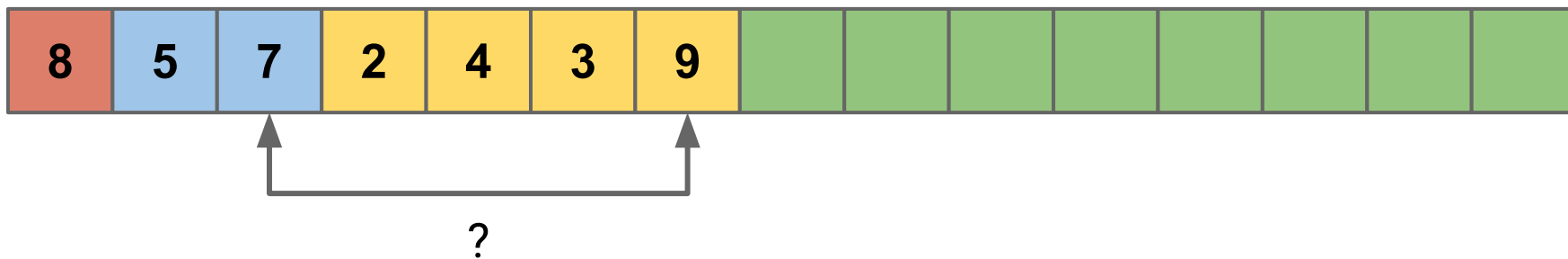
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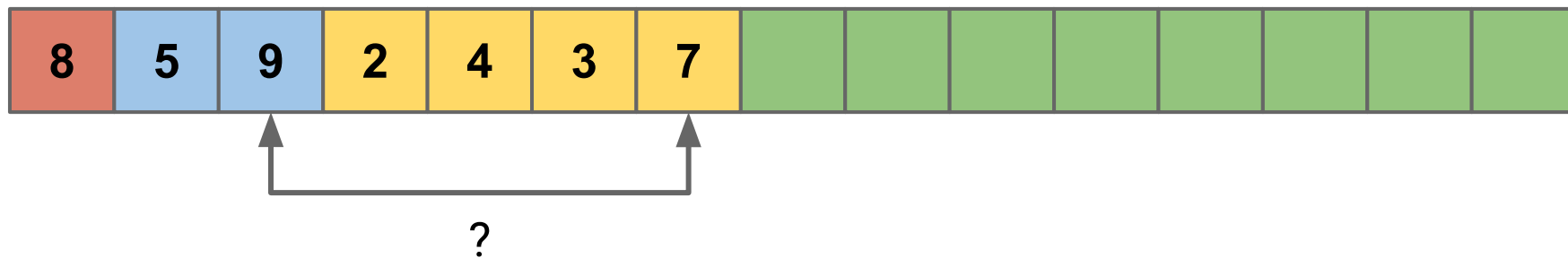
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Heap Sort

1. Insert items into heap
2. Reconstruct sequence (in reverse order) with dequeue

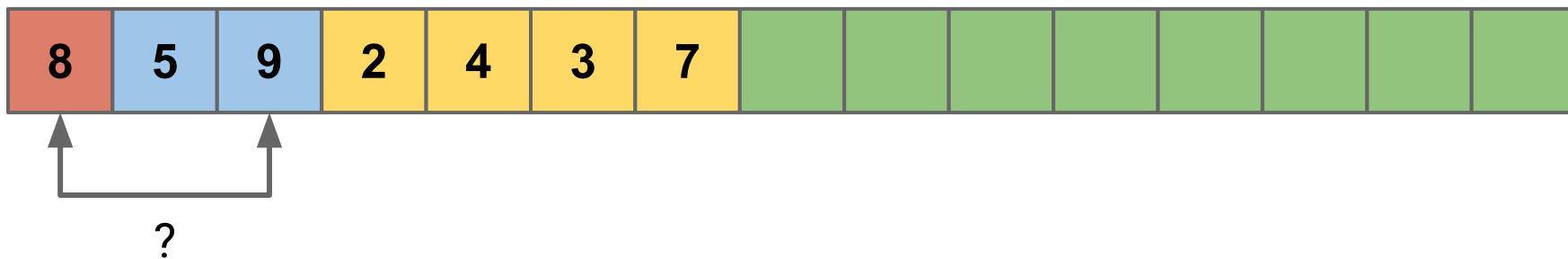
7, 4, 8, 2, 5, 3, 9



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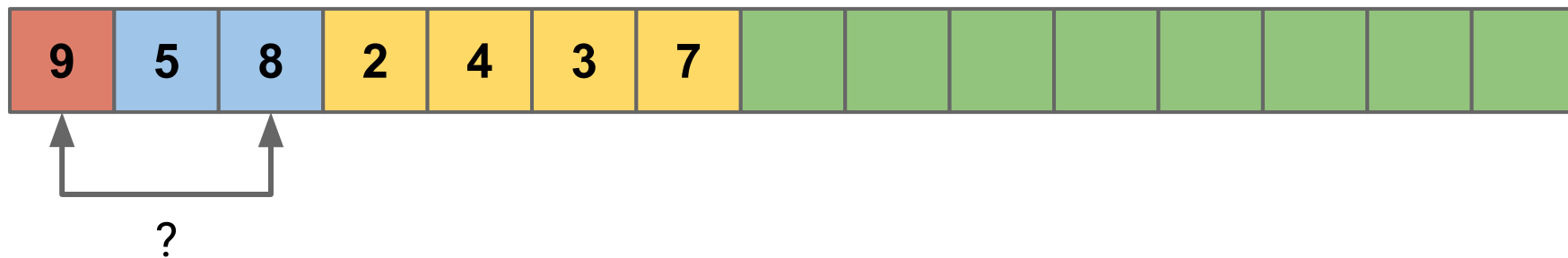
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2, 3, 4, 5, 7, 8, 9

Heap Sort

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Enqueue element i : $O(\log(i))$

Heap Sort

Enqueue element i : $O(\log(i))$

Dequeue element i : $O(\log(n - i))$

Heap Sort

Enqueue element i : $O(\log(i))$

Dequeue element i : $O(\log(n - i))$

$$\left(\sum_{i=1}^n O(\log(i)) \right) + \left(\sum_{i=1}^n O(\log(n - i)) \right)$$

Heap Sort

Enqueue element i : $O(\log(i))$

Dequeue element i : $O(\log(n - i))$

$$\left(\sum_{i=1}^n O(\log(i)) \right) + \left(\sum_{i=1}^n O(\log(n - i)) \right) < O(n \log(n))$$

Updating Heap Elements

What if we want to update a value in our Heap?

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What if we want to update a value in our Heap?

After update we can just call `fixUp` or `fixDown` based on the new value

Updating Heap Elements

What if we want to update a value in our Heap?

After update we can just call `fixUp` or `fixDown` based on the new value

Can we apply this idea to an entire array?

Heapify

Input: Array

Output: Array re-ordered to be a heap

Heapify

Heapify

At level $\log(n)$: Call `fixDown` on all $n/2$ nodes at this level (max 0 swaps each)

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At level $\log(n)-1$: Call `fixDown` on all $n/4$ nodes at this level (max 1 swaps each)

Heapify

At level $\log(n)$: Call `fixDown` on all $n/2$ nodes at this level (max 0 swaps each)

At level $\log(n)-1$: Call `fixDown` on all $n/4$ nodes at this level (max 1 swaps each)

At level $\log(n)-2$: Call `fixDown` on all $n/8$ nodes at this level (max 2 swaps each)

Heapify

At level $\log(n)$: Call `fixDown` on all $n/2$ nodes at this level (max 0 swaps each)

At level $\log(n)-1$: Call `fixDown` on all $n/4$ nodes at this level (max 1 swaps each)

At level $\log(n)-2$: Call `fixDown` on all $n/8$ nodes at this level (max 2 swaps each)

...

At level 1: Call `fixDown` on all 1 nodes at this level (max $\log(n)$ swaps each)

Heapify

Sum the number of swaps
required by each level

$$O \left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i + 1) \right)$$

Heapify

Pull out the n as a constant and distribute multiplication

$$O \left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i + 1) \right)$$

$$O \left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i} \right)$$

Heapify

Focus on the dominant term only

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i}\right)$$

Heapify

Change $\log(n)$ to infinity
(can only increase
complexity class if
anything)

$$O\left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i+1)\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i}\right)$$

$$O\left(n \sum_{i=1}^{\infty} \frac{i}{2^i}\right)$$

Heapify

We can now treat the sum as a constant

$$O \left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i + 1) \right)$$

$$O \left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i} \right)$$

$$O \left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} \right)$$

$$O \left(n \sum_{i=1}^{\infty} \frac{i}{2^i} \right)$$

This is known to converge to a constant

Heapify

Therefore we can heapify
an array of size n in $O(n)$

$$O \left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i + 1) \right)$$

$$O \left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i} \right)$$

$$O \left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} \right)$$

$$O \left(n \sum_{i=1}^{\infty} \frac{i}{2^i} \right) = O(n)$$

Heapify

Therefore we can heapify
an array of size n in $O(n)$

(but heap sort still
requires $n \log(n)$ due to
dequeue costs)

$$O \left(\sum_{i=1}^{\log(n)} \frac{n}{2^i} \cdot (i + 1) \right)$$

$$O \left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} + \frac{1}{2^i} \right)$$

$$O \left(n \sum_{i=1}^{\log(n)} \frac{i}{2^i} \right)$$

$$O \left(n \sum_{i=1}^{\infty} \frac{i}{2^i} \right) = O(n)$$

Next time...

Sets, Bags, and Binary Search Trees...